RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. THIRD SEMESTER EXAMINATION, DECEMBER 2018
SECOND YEAR [BATCH 2017-20]
MATHEMATICS [Honours]
Paper : III

Date : 15/12/2018
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each Group]

Group – A

Answer any five questions from Question Nos. 1 to 8 : [5×7]

1. a) Let $V=\mathbb{C}^4(\mathbb{C})$ and W be a subspace of V generated by $\{(1,2,i,3),(i,3,1,5)\}$. Find a basis for $\frac{V}{W}$.

   b) Find a basis for the row space of $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 3 & 4 & 1 & 2 \\ 0 & 3 & 1 & 1 \\ 5 & 5 & 4 & 7 \end{bmatrix}$. [4+3]

2. a) Let V be a vector space over the field of complex numbers, and suppose that there is an isomorphism T of V onto $\mathbb{C}^3(\mathbb{C})$. Let $\alpha_1,\alpha_{2,}\alpha_3,\alpha_4$ be vectors in V such that $T(\alpha_1)=(1,0,i)$, $T(\alpha_2)=(-2,1+i,0)$, $T(\alpha_3)=(-1,1,1)$, $T(\alpha_4)=(\sqrt{2},i,3)$.

   i) Is α_1 in the subspace spanned by α_2 and α_3? Justify your answer.

   ii) Let W_1 be the subspace spanned by α_1 and α_2 and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2? Justification needed.

   iii) Find a basis for the subspace of V spanned by the four vectors $\alpha_1,\alpha_{2,}\alpha_3$ and α_4. [1 + 2 + 2]

   b) Check wheather $A=\begin{pmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 5 \end{pmatrix}$ is positive definite. [2]

3. a) Find a basis and the dimension of the solution space of the homogeneous system of equations

$$2x_1+2x_2-x_3+x_5=0$$
$$-x_1-x_2+2x_3-3x_4+x_5=0$$
$$x_1+x_2-2x_3-x_5=0$$
$$x_3+x_4+x_5=0$$ [5]

   b) Find two linear operators T and U on $\mathbb{R}^3$ such that TU = O but UT ≠ O, where O represents the zero operator on $\mathbb{R}^3$. [2]

4. a) Let V be a two dimensional vector space over the field F, and let B be an ordered basis of V. If T is a linear operator on V and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in F$, is the matrix of T relative to the ordered basis B, then prove that $T^2 - (a+d)T + (ad-bc)I = O$, where O represents the zero operator on V and I represents the identity operator on V. [4]

b) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y) = (-y,x), \forall (x,y) \in \mathbb{R}^2$. Prove that for every real number c, the operator (T - cI) is invertible, where I is the identity operator on $\mathbb{R}^2$. [3]

5. a) Obtain a non-singular transformation that will reduce the quadratic form $x_1^2 + 2x_1x_2 + 2x_2x_3$ into a normal form. [4]

b) Let V be the vector space of all 2×2 real matrices over $\mathbb{R}$ and $M = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. Let $T: V \to V$ be the linear map defined by T(A) = AM. Find a basis for Ker T. Also find rank of T. [2 + 1]

6. a) Solve the system of equations

$$-x_1 + x_2 + x_3 = a$$

$$x_1 - x_2 + x_3 = b$$

$$x_1 + x_2 - x_3 = c$$

and use the solution to find the inverse of $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. [5]

b) Give an example of a linear operator T on $\mathbb{R}^3$ such that $T^2 = O$ but $T \neq O$. [2]

7. a) V and W are two vector spaces over the same field F of dimensions n and m respectively. Prove that the space L(V,W) has dimension mn. [5]

b) Let T be a linear operator on $\mathbb{R}^3$, the matrix of which in the standard ordered basis is $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$. Find a basis for the range of T. [2]

8. a) Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation, defined by

$T(x,y,z) = (3x+z, -2x+y, -x+2y+4z), \forall (x,y,z) \in \mathbb{R}^3$. Show that T is invertible and determine T^{-1}. [3 + 2]

b) Let $V=\mathbb{R}^4$ and W be a subspace of V generated by {(1, 2, 3, 4), (1, 0, 2, 1)}.Verify that $\dim\frac{V}{W}=\dim V-\dim W$. [2]

Answer <u>any three</u> questions from <u>Question no. 9 to 13</u> : [3 × 5]

9. Let $I\subseteq\mathbb{R}$ be an interval and a sequence of functions $\{f_n\}$ be uniformly convergent on I to a function f. Let f_n be continuous for all $n\in\mathbb{N}$. Show that f is continuous on I. Give an example to show that 'uniform convergence' cannot be replaced by 'pointwise convergence' in the above result. [3 + 2]

10. State Dini's theorem on uniform convergence of sequence of functions. Using Dini's theorem, prove that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is uniformly convergent on [0,1], where $f_1(x)=\sqrt{x}$, $x\in[0,1]$ and $f_{n+1}(x)=\sqrt{xf_n(x)}$ for all $n\geq 1, x\in[0,1]$. [1 + 4]

11. a) State and prove Weierstrass M-test in connection with the uniform convergence of a series of real functions defined on a set $D\subseteq\mathbb{R}$.

    b) Prove that the series $1-\frac{e^{-2x}}{2^2-1}+\frac{e^{-4x}}{4^2-1}-\frac{e^{-6x}}{6^2-1}+\ldots$

    is uniformly convergent on $[0,\infty)$ [3 + 2]

12. a) State Weierstrass Approximation Theorem (on approximation of a real valued continuous function f on [a,b] by a sequence of polynomials).

    Show that it is enough to prove the case a = 0, b = 1 with f(0) = f(1) = 0.

    b) Prove that for every interval [-a,a] , there is a sequence of real polynomials $\{P_n\}$ such that $P_n(0)=0$ and such that $\lim\limits_{n\to\infty}P_n(x)=|x|$ uniformly on [-a,a]. [(1 + 2) + 2]

13. a) Let $\sum\limits_{n=0}^{\infty}a_nx^n$ be a power series and let $\mu=\lim\left|\frac{a_{n+1}}{a_n}\right|$ with $0<\mu<+\infty$.

    Show that the series is absolutely convergent for all x satisfying $|x|<\frac{1}{\mu}$ and the series is divergent for all x satisfying $|x|>\frac{1}{\mu}$.

    b) Determine the radius of convergence of the power series $\frac{1}{3}-x+\frac{x^2}{3^2}-x^3+\frac{x^4}{3^4}-x^5+\ldots$ [3 + 2]

Group – B

Answer any four questions from Question Nos. 14 to 19 : [4×5]

14. A variable plane at a constant distance p from the origin meets the axes at A, B, C. Show that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$. [5]

15. A cone has vertex (0, 0, d) and base (guiding curve) $y^2=4ax, z=0$. Prove that the equation of the reciprocal cone of this cone is $ay^2+dx(z-d)=0$. [5]

16. Find the equation of the sphere which passes through the circle $y=0,(x-a)^2+(z-c)^2=r^2$ and touches the plane x = 0. Show that the area which it cuts off from the plane z = 0 is $\pi(a^2-c^2)$. [5]

17. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two. [5]

18. Find the locus of the point of intersection of the perpendicular generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. [5]

19. Reduce the equation $2x^2-7y^2+2z^2-10yz-8zx-10xy+6x+12y-6z+5=0$ by orthogonal transformation to the canonical form and determine the nature of the quadric. [4+1]

Answer any two questions from Questions nos. 20 to 22: [2×12]

20. a) A particle of mass m is acted on by a force mn^2x to a fixed point when at a distance x from it, and also by a force mP cos (2nt) in the line joining it to the fixed point. Initially it is at rest at the point. Prove that in the subsequent motion its greatest displacement on one side of the fixed point is $\frac{3P}{8n^2}$, and on the other side $\frac{2P}{3n^2}$. [7]

    b) A pair of rectangular axes are rotating in their plane with a uniform angular velocity ‘ω’. Find the expressions of the velocity and acceleration of a moving particle referred to the rotating axes. [5]

21. a) A particle is launched at an angle α from a cliff of height H above sea-level. If it falls into the sea at a distance D from the base of the cliff, prove that its maximum height above the sea-level is $H+\frac{D^2\tan^2\alpha}{4(H+D\tan\alpha)}$. [6]

    b) A particle of mass m is attached to a fixed point by an elastic string of natural length ‘a’, the coefficient of elasticity being ‘nmg’. It is projected from an apse at a distance ‘a’ with velocitly $\sqrt{2pgh}$. Prove that the other apsidal distance is given by the equation $nr^2(r-a)-2pha(r+a)=0$ [6]

22. a) If a particle is projected from an apse at a distance 'a' with the velocity from infinity under the action of a central acceleration μu^{2n+3},prove that the path is $r^n = a^n \cos n\theta$. [6]

b) One end of an elastic string of unstretched length 'a' is tied to a point on the top of a smooth table and a particle attached to the other end can move freely on the table. If the path be nearly a circle of radius 'b', prove that its apsidal angle is approximately $\pi\sqrt{\frac{b-a}{4b-3a}}$. [6]

Answer <u>any one</u> question from question <u>nos. 23 to 24</u> : [1×6]

23. A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{2gh}$; prove that the latus rectum of a parabola described after leaving the arc is $\frac{h^2}{2a}$, where a is the radius of the generating circle. [6]

24. A car of mass m starts from rest and moves on a level road under a constant frictional resistance, the engine working at a constant rate P. If the maximum speed is V and the speed u is attained after travelling a distance 's' in time 't', show that $t=\frac{s}{V}+\frac{mu^2}{2P}$. [6]

——— × ———